Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(99)00024-6
J. Appl. Maths Mechs, Vol. 63, No. 2, pp. 165–171, 1999

0021-8928/99/$—see front matter

FORMS OF HAMILTON'S PRINCIPLE IN QUASI-COORDINATES†

V. V. RUMYANTSEV

Moscow

(*Received* 3 *September* 1998)

In a previous paper [1], certain conditions, due to Hölder, Voronets and Suslov in the case of linear constraints, for deriving three forms of Hamilton's principle in generalized coordinates and velocities for the general case of non-linear non-holonomic constraints were analysed. It was shown that these three forms are equivalent and transform into one another. As a sequel to that analysis, similar issues are investigated for the case of non-linear quasi-coordinates and quasi-velocities and, in addition, the three forms of Hamilton's principle are exhibited in the case of a Legendre transformation, which transforms the equation of motion to canonical form in quasi-coordinates. © 1999 Elsevier Science Ltd. All rights reserved.

1. We first consider a holonomic system with Lagrange coordinates q_i and velocities $\dot{q}_i$ subject to forces with a force function $U(t, q_1, \ldots, q_n)$, and arbitrary independent smooth functions

$$\eta_i \equiv f_i(t, q, \dot{q}) \tag{1.1}$$

which are generally non-linear in the generalized velocities and such that $\det(\partial f_i/\partial \dot{q}_j) \neq 0$. Throughout Section 1, the indices i, j, r and s take the values $1, \ldots, n$.

Solving relations (1.1), we obtain the expressions

$$\dot{q}_i = F_i(t, q, \eta) \tag{1.2}$$

substitution of which into (1.1) makes the latter identities.

Obviously,

$$f_{si}F_{ir} = \delta_{sr}, \quad f_{ir}F_{si} = \delta_{rs} \tag{1.3}$$

where, as always below, summation is to be performed over repeated indices, and moreover $f_{si} \equiv \partial f_s/\partial \dot{q}_i$, $F_{ir} \equiv \partial F_i/\partial \eta_r$; δ_{sr} is the Kronecker delta.

Following Hamel [2], we introduce the notation $\dot{\pi}_s = \eta_s$, where π_s and η_s are non-linear quasi-coordinates and quasi-velocities, and moreover

$$\frac{\partial}{\partial \pi_s} \equiv F_{is}\frac{\partial}{\partial q_i}, \quad \frac{\partial}{\partial q_i} = f_{si}\frac{\partial}{\partial \pi_s} \tag{1.4}$$

Virtual displacements in Lagrange coordinates and quasi-coordinates satisfy the relationships

$$\delta q_i \equiv F_{is}\delta\pi_s, \quad \delta\pi_s \equiv f_{si}\delta q_i \tag{1.5}$$

Substituting expressions (1.2) into the Lagrangian $L(t, q, \dot{q}) = T(t, q, \dot{q}) + U(t, q)$, where $T(t, q, \dot{q})$ is the kinetic energy of the system, we obtain a function $L^*(t, q, \eta)$, in terms of which the D'Alembert–Lagrange principle takes the following form in quasi-coordinates

$$\left(\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s} + \frac{\partial L^*}{\partial \eta_r}W_s^r - \frac{\partial L^*}{\partial \pi_s}\right)\delta\pi_s = 0 \tag{1.6}$$

or the following form

†*Prikl. Mat. Mekh.* Vol. 63, No. 2, pp. 172–178, 1999.

$$\left(\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}-\frac{\partial L^*}{\partial \eta_r}f_{ri}T_s^i-\frac{\partial L^*}{\partial \pi_s}\right)\delta\pi_s=0 \tag{1.7}$$

where we have used the notation of [3]

$$W_s^r\equiv F_{is}\left(\frac{df_{ri}}{dt}-\frac{\partial f_r}{\partial q_i}\right),\ T_s^i\equiv\frac{dF_{is}}{dt}-\frac{\partial F_i}{\partial \pi_s} \tag{1.8}$$

Since $\delta\pi_s$ are arbitrary, it follows from Eq. (1.6) or Eq. (1.7) that the equations of motion of a holonomic system under the action of potential forces in quasi-coordinates are

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}+\frac{\partial L^*}{\partial \eta_r}W_s^r-\frac{\partial L^*}{\partial \pi_s}=0 \tag{1.9}$$

or

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}-\frac{\partial L^*}{\partial \eta_r}f_{ri}T_s^i-\frac{\partial L^*}{\partial \pi_s}=0 \tag{1.10}$$

Equations of the form (1.9) or (1.10) were first derived by Hamel [2] from the central Lagrange equation using the following transitivity equations, which he established on the assumption that $d\delta q_i=\delta dq_i$

$$\frac{d\delta\pi_s}{dt}-\delta\eta_s=W_r^s\delta\pi_r \quad\text{and}\quad \frac{d\delta\pi_s}{dt}-\delta\eta_s=-\frac{\partial\eta_s}{\partial\dot q_i}T_r^i\delta\pi_r \tag{1.11}$$

Equations (1.9) and (1.10) were referred to as the first and second forms of the equations of motion, respectively. Hamel pointed out a disadvantage of Eqs (1.9): calculation of the coefficients W_s^r involves $\dot q_i$ and $\ddot q_i$, while Eqs (1.10) involve functions f_{ri}, which depend on $\dot q_j$, which may be expressed in terms of functions which depend on q_i and η_s. Equations (1.10) were a natural generalization of the Lagrange–Euler equations in linear quasi-coordinates, which had previously been derived by Hamel [2].

Novoselov [3], also using the transitivity equations (1.11), deduced Eqs (1.9) and (1.10) (with the factors $(\partial L^*/\partial\eta_r)f_{ri}$ in the latter replaced by $\partial L/\partial\dot q_i$) from Hamilton's principle

$$\delta\int_{t_0}^{t_1}L^*dt=0,\ \delta\pi_s\big|_{t_0}=\delta\pi_s\big|_{t_1}=0 \tag{1.12}$$

and called them equations of Voronets–Hamel and Chaplygin type, respectively.

The same equations have been derived from the Maggi equation without using Eqs (1.11) [4].

Note that Eqs (1.9) or (1.10), in turn, imply Hamilton's principle (1.12); to verify this, one need only multiply them by $\delta\pi_s$, sum over s, integrate the result with respect to t from t_0 to t_1, and then use (1.11).

Using the Legendre transformation [5]

$$y_s=\frac{\partial L^*}{\partial\eta_s},\quad H^*(t,q,y)=y_s\eta_s-L^*(t,q,\eta) \tag{1.13}$$

We were able [4] to reduce Eqs (1.9) and (1.10) to the canonical form of equations in quasi-coordinates:

$$\frac{dy_s}{dt}+y_rW_s^r+\frac{\partial H^*}{\partial\pi_s}=0,\ \eta_s=\frac{\partial H^*}{\partial y_s} \tag{1.14}$$

or

$$\frac{dy_s}{dt}-y_rf_{ri}T_s^i+\frac{\partial H^*}{\partial\pi_s}=0,\ \eta_s=\frac{\partial H^*}{\partial y_s} \tag{1.15}$$

The coefficients W_s^r and T_s^r in these equations must be expressed in terms of y_r using relationships (1.1), (1.2) and (1.13).

Equations (1.14) or (1.15) enable one to derive the second form of Hamilton's principle. Indeed, we multiply the first group of equations (1.14) or (1.15) by $\delta\pi_s$ and the second group by $-\delta y_s$, sum over all s, and integrate the result with respect to t; then, using (1.11) and putting $\delta\pi_s = 0$ at $t = t_0, t_1$, we obtain

$$0 = \int_{t_0}^{t_1}\left(\left(\frac{dy_s}{dt} + y_r W_s^r + \frac{\partial H^*}{\partial \pi_s}\right)\delta\pi_s - \left(\eta_s - \frac{\partial H^*}{\partial y_s}\right)\delta y_s\right)dt =$$

$$= -\int_{t_0}^{t_1}\left(y_s\left(\frac{d\delta\pi_s}{dt} - W_r^s\delta\pi_r\right) + \eta_s\delta y_s - \delta H^*\right)dt = -\delta\int_{t_0}^{t_1}(y_s\eta_s - H^*)dt$$

We have thus proved the validity of the second form of Hamilton's principle in quasi-coordinates for a holonomic system:

$$\delta\int_{t_0}^{t_1}(y_s\eta_s - H^*)dt = 0, \quad \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0 \tag{1.16}$$

Equations (1.14) and (1.15), in turn, may be derived from principle (1.16). It should be noted that principle (1.16) is important in its own right, in view of the assumption that the variation δy_s are arbitrary and independent of the virtual displacement $\delta\pi_s$ inside the interval (t_0, t_1) [5].

2. We will now consider a non-holonomic system with k degrees of freedom, subject to non-linear constraints of the form

$$\eta_\alpha \equiv f_\alpha(t,q,\dot q) = 0, \quad \operatorname{rank}(f_{\alpha i}) = n - k$$
$$i = 1,\ldots, n; \ \alpha = k+1,\ldots, n \tag{2.1}$$

Let us introduce arbitrary quasi-velocities

$$\eta_s = \frac{d\pi_s}{dt} \equiv f_s(t,q,\dot q), \quad s = 1,\ldots,k \tag{2.2}$$

such that $\det(f_{si}) \neq 0$ $(i, s = 1, \ldots, n)$, so that equalities (2.1) and (2.2) yield expressions of the type (1.2). The virtual displacements of a non-holonomic system are defined by Chetayev's conditions

$$\frac{\partial f_\alpha}{\partial \dot q_i}\delta q_i = 0, \quad \alpha = k+1,\ldots,n \tag{2.3}$$

As before, the virtual displacements satisfy relations (1.5), but only for $s = 1, \ldots, k$, since, by (2.3), constraints (2.1) imply that $\delta\pi_\alpha = 0$ $(\alpha = k+1, \ldots, n)$, while the quantities $\delta\pi_s$ $(s = 1, \ldots, k)$ are arbitrary. Under these conditions, the D'Alembert–Lagrange principle (1.6) or (1.7) implies the equations of motion of the non-holonomic system in quasi-coordinates in the form (1.9) or (1.10), but only for $s = 1, \ldots, k$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s} + \frac{\partial L^*}{\partial \eta_r}W_s^r - \frac{\partial L^*}{\partial \pi_s} = 0, \quad \eta_\alpha = 0 \tag{2.4}$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s} - \frac{\partial L^*}{\partial \eta_r}f_{ri}T_s^i - \frac{\partial L^*}{\partial \pi_s} = 0, \quad \eta_\alpha = 0 \tag{2.5}$$

$$s = 1,\ldots, k; \ \alpha = k+1,\ldots, n; \ i, r = 1,\ldots, n$$

One can set $\eta_\alpha = 0$ in these equations only after expressing them in explicit form, since they generally involve all the derivatives $\partial L^*/\partial\eta_r$ $(r = 1, \ldots, n)$.†

The first k of both groups of the transitivity equations (1.11) retain their form provided that one has $\delta\pi_r = 0$ for $r = k + 1, \ldots, n$ on their right-hand sides, while the remaining equations take the following form [3]

$$\delta\eta_\alpha = -W_r^\alpha \delta\pi_r \text{ and } \delta\eta_\alpha = \frac{\partial\eta_\alpha}{\partial\dot{q}_i} T_r^i \delta\pi_r \tag{2.6}$$

$$r = 1,\ldots, k;\ \alpha = k + 1,\ldots, n;\ i = 1,\ldots, n$$

Similarly, the canonical equations of motion have the form of the first k pairs of Eqs (1.14) and (1.15), that is, for $s = 1, \ldots, k$, to which one must add the equations of constraint (2.1) rewritten as

$$\partial H^* / \partial y_\alpha = 0,\ \alpha = k+1,\ldots,n \tag{2.7}$$

For a non-holonomic system, Hamilton's principle in the form (1.12) or (1.16) is generally not valid, since indirect paths do not satisfy the equations of constraint (2.1) [1, 6]; but it is valid in Hölder's form

$$\int_{t_0}^{t_1} \delta L^* dt = 0,\ \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0,\ s = 1,\ldots,k \tag{2.8}$$

and also in the corresponding second form

$$\int_{t_0}^{t_1} \delta(y_s\eta_s - H^*)dt = 0,\ \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0,\ s = 1,\ldots,k \tag{2.9}$$

We will now derive the Voronets equations in quasi-coordinates, which involve terms which depend on the kinetic energy of the non-holonomic system [7]. To do this, we replace the kinetic energy $T^*(t, q, \eta)$ of a holonomic system, which occurs in the function $L^*(t, q, \eta)$ in Eqs (2.4) and (2.5), by the kinetic energy $\Theta^*(t, q, \eta_1, \ldots, \eta_k)$ of a non-holonomic system with constraints (2.1). Since the following relationships hold when $\eta_\alpha = 0$ $(\alpha = k + 1, \ldots, n)$ [8]

$$\frac{\partial L^*}{\partial\eta_s} = \frac{\partial\Theta^*}{\partial\eta_s},\ \frac{\partial L^*}{\partial\pi_s} = \frac{\partial(\Theta^* + U)}{\partial\pi_s},\ \frac{\partial L^*}{\partial\eta_\alpha} = \left(\frac{\partial T^*}{\partial\eta_\alpha}\right)_0;\ s = 1,\ldots,k;\ \alpha = k+1,\ldots,n \tag{2.10}$$

where

$$\left(\frac{\partial T^*}{\partial\eta_\alpha}\right)_0 = \frac{\partial T^*}{\partial\eta_\alpha}\bigg|_{\eta_\beta = 0},\ \beta = k+1,\ldots,n$$

it follows that the first k equations of (2.4) and (2.5) may be expressed in the form of the Voronets equations in quasi-coordinates

$$\frac{d}{dt}\frac{\partial\Theta^*}{\partial\eta_s} - \frac{\partial(\Theta^* + U)}{\partial\pi_s} + \left(\frac{\partial T^*}{\partial\eta_i}\right)_0 W_s^i = 0,\ s = 1,\ldots,k;\ i = 1,\ldots,n \tag{2.11}$$

and

$$\frac{d}{dt}\frac{\partial\Theta^*}{\partial\eta_s} - \frac{\partial(\Theta^* + U)}{\partial\pi_s} - \left(\frac{\partial T}{\partial\dot{q}_i}\right)^* T_s^i = 0,\ s = 1,\ldots,k;\ i = 1,\ldots,n \tag{2.12}$$

†We take this opportunity to correct some misprint in [4]: on p. 538, line 11 from the top, "$r, s = 1, \ldots, k$" should read: "$s = 1, \ldots, k$"; in line 13 from the top, "(1) and (11)" should read: "(I) and (II)". [These misprints were corrected in the English translation of [4] (Ed.).]

where $(\partial T/\partial \dot{q}_i)^*$ denotes the result of replacing $\dot{q}_i$ by (1.2) in the expression $\partial T/\partial \dot{q}_i$ $(i = 1, \ldots, n)$.

Equations (2.11) and (2.12) imply the Voronets form of Hamilton's principle for a non-holonomic system, in the form

$$\int_{t_0}^{t_1}\left(\delta(\Theta^* + U) - \left(\frac{\partial T^*}{\partial \eta_\alpha}\right)_0 W_s^\alpha \delta\pi_s\right)dt = 0, \quad \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0 \tag{2.13}$$

or in the form

$$\int_{t_0}^{t_1}\left(\delta(\Theta^* + U) + \left(\frac{\partial T}{\partial \dot{q}_i}\right)^* T_s^i \delta\pi_s\right)dt = 0, \quad \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0 \tag{2.14}$$

Equations (2.11) or (2.12), in turn, may be deduced from (2.13) or (2.14), respectively [6].

Using the Legendre transformation

$$y_s = \frac{\partial \Theta^*}{\partial \eta_s}, \quad H^*(t, q, y) = y_s\eta_s - \Theta^*(t, q, \eta) - U(t, q), \quad s = 1, \ldots, k \tag{2.15}$$

subject to the condition $\| \partial^2\Theta^*/\partial\eta_r\partial\eta_s \| \neq 0$ $(r, s = 1, \ldots, k)$, one can reduce Eqs (2.11) and (2.12) to the canonical form of equations in quasi-coordinates

$$\frac{dy_s}{dt} + \frac{\partial H^*}{\partial \pi_s} + \left(\frac{\partial T^*}{\partial \eta_i}\right)_0 W_s^i = 0, \quad \eta_s = \frac{\partial H^*}{\partial y_s} \tag{2.16}$$

and

$$\frac{dy_s}{dt} + \frac{\partial H^*}{\partial \pi_s} - \left(\frac{\partial T}{\partial \dot{q}_1}\right)^* T_s^i = 0, \quad \eta_s = \frac{\partial H^*}{\partial y_s} \tag{2.17}$$

Equations (2.16) and (2.17) enable one to obtain the second Voronets form of Hamilton's principle, in the form

$$\int_{t_0}^{t_1}\left(\delta(y_s\eta_s - H^*) - \left(\frac{\partial T^*}{\partial \eta_\alpha}\right)_0 W_s^\alpha \delta\pi_s\right)dt = 0, \quad \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0 \tag{2.18}$$

or in the form

$$\int_{t_0}^{t_1}\left(\delta(y_s\eta_s - H^*) + \left(\frac{\partial T}{\partial \dot{q}_i}\right)^* T_s^i \delta\pi_s\right)dt = 0, \quad \delta\pi_s\big|_{t_0} = \delta\pi_s\big|_{t_1} = 0 \tag{2.19}$$

$s = 1, \ldots, k;\ i = 1, \ldots, n;\ \alpha = k + 1, \ldots, n.$

Using relations (2.10), it is not difficult to verify the truth of the equalities

$$\delta L^* = \delta(\Theta^* + U) - \left(\frac{\partial T}{\partial \eta_\alpha}\right)_0^* W_s^\alpha \delta\pi_s = \delta(\Theta^* + U) + \left(\frac{\partial T^*}{\partial \dot{q}_i}\right) T_s^i \delta\pi_s \tag{2.20}$$

which prove that the Voronets forms of Hamilton's principle in quasi-coordinates for a non-holonomic system are equivalent to the Hölder forms.

In conclusion, we consider an important special case in which the non-holonomic constraints (2.1) are solved with respect to certain generalized velocities. Suppose constraints (2.1) are given in the following form [1]

$$\eta_\alpha \equiv f_\alpha(t,q,\dot q) = \dot q_\alpha - \varphi_\alpha(t,q,\dot q_1,\dots,\dot q_k) = 0 \tag{2.21}$$

As parameters η_s, we take the independent velocities $\dot q_s$:

$$\eta_s = \dot q_s \tag{2.22}$$

Throughout, $\alpha = k+1,\dots,n; s = 1,\dots,k$.

In this case, Eqs (2.6) become the equations [1]

$$\delta\eta_\alpha = \delta\dot q_\alpha - \delta\varphi_\alpha = A_s^\alpha \delta q_s \tag{2.23}$$

and the first k transitivity equations (1.1) become identifies, since

$$W_r^s = -\frac{\partial \eta_s}{\partial \dot q_i} T_r^i = 0, \quad r = 1,\dots,k$$

and moreover

$$A_s^\alpha = \frac{d}{dt}\frac{\partial\varphi_\alpha}{\partial \dot q_s} - \frac{\partial\varphi_\alpha}{\partial q_s} - \frac{\partial\varphi_\alpha}{\partial q_\beta}\frac{\partial\varphi_\beta}{\partial \dot q_s}, \quad \beta = k+1,\dots,n \tag{2.24}$$

The Voronets equations (2.11) take the form [7, 1]

$$\frac{d}{dt}\frac{\partial\Theta}{\partial \dot q_s} - \frac{\partial(\Theta+U)}{\partial q_s} - \frac{\partial(\Theta+U)}{\partial q_\alpha}\frac{\partial\varphi_\alpha}{\partial \dot q_s} - \frac{\partial T}{\partial \dot q_a} A_s^\alpha = 0 \tag{2.25}$$

where $\Theta(t, q, \dot q_1, \dots, \dot q_k)$ is the kinetic energy of a non-holonomic system in generalized velocities, into which the kinetic energy of a holonomic system $T(t, q, \dot q)$ is transformed when constraints (2.21) are taken into account. Equations (2.25) were derived by Voronets [7] from Hamilton's principle in the Voronets form†

$$\int_{t_0}^{t_1}\left(\delta(\Theta+U) + \frac{\partial T}{\partial \dot q_\alpha}(\delta\dot q_\alpha - \delta\varphi_\alpha)\right)dt = 0, \quad \delta q_s\big|_{t_0} = \delta q_s\big|_{t_1} = 0 \tag{2.26}$$

which is equivalent to the Hölder form by virtue of (2.20), the latter taking the following form in terms of generalized coordinates and velocities [1]

$$\delta T = \delta\Theta + \frac{\partial T}{\partial \dot q_\alpha}(\delta\dot q_\alpha - \delta\varphi_\alpha) \tag{2.27}$$

Using the Legendre transformation

$$p_s = \frac{\partial\Theta}{\partial \dot q_s}, \quad H(t,q,p) = p_s\dot q_s - \Theta(t,q,\dot q_s) - U(t,q)$$

we reduce Eqs (2.25) to canonical form

$$\frac{dp_s}{dt} + \frac{\partial H}{\partial q_s} + \frac{\partial H}{\partial q_\alpha}\left(\frac{\partial\varphi_\alpha}{\partial \dot q_s}\right)^* - \left(\frac{\partial T}{\partial \dot q_\alpha}A_s^\alpha\right)^* = 0, \quad \frac{\partial q_s}{\partial t} = \frac{\partial H}{\partial p_s} \tag{2.28}$$

where $(\psi)^*$ denotes the expression of ψ in terms of p_s.

†Note that in [6] formula (51), equivalent to (2.26), and Section 6 were incorrectly referred to as Suslov's principle; the latter has the form (2.31) (see below).

Equations (2.28) lead to the second Voronets form of Hamilton's principle in generalized coordinates and momenta

$$\int_{t_0}^{t_1}\left(\delta(p_s\dot{q}_s - H) + \left(\frac{\partial T}{\partial \dot{q}_\alpha}A_s^\alpha\right)^{*}\delta q_s\right)dt = 0,\quad \delta q_s\big|_{t_0} = \delta q_s\big|_{t_1} = 0 \tag{2.29}$$

Hölder and Voronets assumed that $\delta dq_i = d\delta q_i$ ($i = 1, \ldots, n$). According to another point of view, due to Appell and Suslov, such relations only hold for $i = 1, \ldots, k$, while for $\alpha = k + 1, \ldots, n$ one has

$$\frac{d}{dt}\delta q_\alpha - \bar{\delta}\dot{q}_\alpha = A_s^\alpha \delta q_s \tag{2.30}$$

where $\bar{\delta}$ denotes the variation in the Appell–Suslov sense. This leads to Suslov's form of Hamilton's principle [9, 1]

$$\int_{t_0}^{t_1}\left(\bar{\delta}L + \frac{\partial T}{\partial \dot{q}_\alpha}A_s^\alpha \delta q_s\right)dt = 0,\quad \delta q_s\big|_{t_0} = \delta q_s\big|_{t_1} = 0 \tag{2.31}$$

Since, on the assumption that (2.30) holds, equality (2.27) becomes $\bar{\delta}T = \delta\Theta$, it is obvious that (2.31) may be reduced to the Voronets form (2.26) of Hamilton's principle.

The research reported here was supported financially by the Russian Foundation for Basic Research (96-01-00261) and the Federal Special-Purpose "Integration" Programme.

REFERENCES

1. RUMYANTSEV, V. V., Hamilton's principle for non-holonomic systems. *Prikl. Mat. Mekh.*, 1978, **42**, 3, 387–399.
2. HAMEL, G., *Theoretische Mechanik*. Springer, Berlin, 1949.
3. NOVOSELOV, V. S., *Variational Methods in Mechanics*. Izd. Leningrad. Gos. Univ., Leningrad, 1966.
4. RUMYANTSEV, V. V., Poincaré's and Chetayev's equations. *Prikl. Mat. Mekh.*, 1998, **62**, 4, 531–538.
5. CHETAYEV, N. G., Poincaré's equations. In *Theoretical Mechanics*. Nauka, Moscow, 1987, 287–322.
6. PAPASTAVRIDIS, J. G., Time-integral variational principles for nonlinear nonholonomic systems. *J. Appl. Mech.*, 1997, **64**, 985–991.
7. VORONETS, P. V., The equations of motion for non-holonomic systems. *Mat. Sbornik*, 1901, **22**, 4, 659–686.
8. RUMYANTSEV, V. V., The Poincaré–Chetayev equations. *Prikl. Mat. Mekh.*, 1994, **58**, 3, 3–16.
9. SUSLOV, G. K., A version of D'Alembert's principle. *Mat. Sbornik*, 1901, **22**, 4, 687–691.

Translated by D.L.